

A Work Project, presented as part of the requirements for the Award of a Master Degree in Management from the NOVA – School of Business and Economics.

“NEGATIVE INTEREST RATES: how the existent financial models can fit with this new scenario? A focus on Vasicek and CIR”

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DATE:

04/05/2017

Negative interest rates: how the existent financial models can fit with this new scenario?

A focus on Vasicek and CIR.

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ABSTRACT

My project work has the purpose to question on the use of financial models with this new scenario of negative interest rates. Precisely, the two models I am going to study are the Vasicek and CIR.

First of all, the research will be focused on the analysis of them looking at both their limitations and strengths, and trying to make some needed and essential modifications for this shift in macroeconomic scenario.

The entire work has the objective of understanding which are the economic agents affected and which are the future perspectives in this economic situation.

KEY WORDS: Negative interest rates, Vasicek and CIR models, Hull and White, New macroeconomic policies

1. Introduction

Karl Popper stated “In so far as a scientific statement speaks about reality, it must be falsifiable, and in so far as it is not falsifiable, it does not speak about reality.”

Negative interest rates have appeared in the last years, they have been set by different central banks, among which ECB, with the aim of helping the economy to restart. However, this possibility was not contemplated in economic and financial handbook, since it was considered a fictitious scenario, inconceivable in the real world. Though, as Popper affirmed scientific statements must be falsifiable to be adapted to reality, otherwise they are not able to continue to work.

The choice to write a thesis on this financial topic, the presence in real world of negative interest rates, is driven by the deep interest towards this field of study and the will to understand if the already existent financial models are able to continue working with this new economic situation, proposing some developments of study. It is important, in my opinion, focusing the attention on this change given its impact on both economic and monetary system and on citizens' everyday life. My work wants to start from the general macroeconomic dynamics, which have forced the principal central banks to overpass their “classic” monetary policies choices, up to, as in the special case of ECB, enlarge its perspective in order to allow the employment of instruments never used before. What has happened after the last economic crisis? Europe has started experimenting a period of heavy deflation, and given that the main objective of ECB is to maintain price stability through the control of inflation, it can operate setting and adjusting interest rates. If from one side, ECB used to increase interest rates in order to fight against too high inflation; on the other side, it can contrast deflation decreasing the rates. This has been the main objective for which negative interest rates have been set. It is

important to keep in mind, however, that sometimes tools considered necessary to heal the economy, as medicines, can have much more serious side effects of the pathology that are called upon to treat. Especially if an error occurs in both the intervention times and in the doses methods of administration, and the virus that we would like to eradicate becomes immune to treatment, forcing the doctor, in the increasingly desperate attempt to save the patient, whether increasing the dosage of drugs or trying to experiment new ones, even more powerful. This vicious circle can lead to death of the virus, such as of the sick.

I would like now explaining how the work is going to be structured.

In the **II** chapter, I am going to make a literature review, with the aim of building a framework of the existent written works. I will explore the Vasicek and the CIR model, assessing the main assumptions and the results provided by them. I went through these two models with the two main papers; which are an equilibrium characterization of the term structure by Oldrich Vasicek, and a theory of the term structure of interest rates by John C. Cox, Jonathan E. Ingersoll, Jr., and Stephen A. Ross.

In the **III** chapter, I will explain the development of the models I have decided to pursue in this work. In the first paragraph, I will go through each step I followed during my research, explaining the procedure and each alteration needed (particularly the study will be divided in three different stages); later on I will focus on the data I used for applying and testing the models; and last but not least I will explain the results obtained, followed by some explanatory graphs.

Finally, in the **IV** and last chapter, I will close the project summing up all the main findings and making a final evaluation of the current scenario. I will go deeper in the future perspectives and I will try to explain both sides of the balance brought by difficult decisions that sometimes must be taken.

2. Literature review

Many theories have been proposed to explain the construction of the term structure, describing the evolution during the time of the entire zero curve. Although the literature covers a wide variety of such theories, this review will focus on two major models, which are equilibrium models. Precisely, they are: an equilibrium characterization of the term structure by Oldrich Vasicek, and a theory of the term structure of interest rates by Cox, Ingersoll, and Ross. Although the literature presents a full explanation of the doctrine, this work will primarily focus on the development of this model in order to fit them to the current scenario of negative interest rates, adopted by the main central banks during the last economic crisis (e.g. ECB fixed for the first time in its history a negative interest rate in June 2014).

2.1 An equilibrium characterization of the term structure by Oldrich Vasicek

“This paper derives a general form of the term structure of interest rates”¹. Before presenting the fundamental assumptions of the model and explain the results obtained, it is necessary to underline and expose some notations on the scenario in which the model operates. Within the construction of the model, it is defined a market where “default free claims” (that are discount bonds on an established amount of money, which can be delivered at a given future date) are traded by investors. It is, also, necessary to determine the key value of the model in order to better understand all the steps of the development. Let’s start with $P(t, s)$, which represents the price at time t of a discount bond maturing at time s , with $t \leq s$ and a unit maturity value $P(s, s) = 1$. Proceeding, $R(t, T)$ is the internal rate of return at time t on a bond with maturity date $s = t + T$.

$$R(t, T) = -\frac{1}{T} \log P(t, t + T) \quad \text{with } T > 0 \quad (1)$$

$F(t, s)$, instead, is defined as the marginal rate of return given by investing in a bond for an additional instant, i.e. the forward rate:

¹ Vasicek, O. A. “An equilibrium characterization of the term structure.” *Journal of Financial Economics*.

$$R(t, T) = \frac{1}{T} \int_t^{t+T} F(t, \tau) d\tau \rightarrow F(t, s) = \frac{\partial}{\partial s} [(s - t)R(t, s - t)] \quad (2)$$

The spot rate, i.e. the instantaneous rate at which is possible to borrow and lend, is identified with $r(t)$.

$$r(t) = R(t, 0) = \lim_{T \rightarrow 0} R(t, T) \quad (3)$$

To conclude, W is the amount of the loan, which is going to be borrowed and lent, and we can define:

$$dW = Wr(t)dt \quad (4)$$

meaning that at any time, the current value $r(t)$ of the spot rate is the value at which the value of the loan increase. Given this scenario, now I would like to put in evidence the three main assumptions on which the model is built. First, the spot rate is determined by a stochastic differential equation in the form of:

$$dr = f(r, t)dt + \rho(r, t)dz \quad (5)$$

where the first term represents the drift and the second the variance. dr follows a continuous Markov process, meaning that the instantaneous interest rate is characterized by a single state variable (i.e. the current value) and that the probability distribution of the segment $\{r(\tau), \tau \geq t\}$ is completely determined by the value of $r(t)$. Continuing, the second assumption is that the price $P(t, s)$ of a discount bond depends on the behavior, at time t , of this just mentioned segment over the term of the bond. Here, what comes out are three main concepts: expectation, market segmentation, and liquidity preference hypotheses, and $R(t, T)$ can be defined as:

$$R(t, T) = E_t \left(\frac{1}{T} \int_t^{t+T} r(\tau) d\tau \right) + \bar{\pi}(t, T, r(t)) \quad (6)$$

Finally, the last and third assumption states that the market is efficient, that is there are no transaction costs, information is available to all investors simultaneously, and evenly investors act rationally (they prefer more wealth to less, and use all available information).

$$P(t, s) = P(t, s, r(t)) \quad (7)$$

Once set this scenario, it is possible to affirm that the value of the spot rate is the only dependent variable for the whole term structure. Also the process of the bond price is determined by:

$$dP = P\mu(t, s)dt - P\sigma(t, s)dz. \text{ }^i \quad (8)$$

In this paper it is also studied the construction of the market price of risk, an essential measure in the use and application of these models. Let's define it $q(t, r)$, which represents how much the instantaneous rate of return on a bond increase with an additional unit of risk. For a bond of any maturity, we can define:

$$q(t, r) = \frac{\mu(t, s, r) - r}{\sigma(t, s, r)} \quad s \geq t \quad (9)$$

which can be used to derive the equation to find the price of a discount bond. To conclude, assumptions 1, 2, and 3, reported above, are used to show that the expected rate of return on any bond in excess of the spot rate is proportional to its standard deviation. "This property is then used to derive a partial differential equation for bond prices"² in the form of:

$$P(t, T) = A(t, T)e^{-B(t, T)r(t)} \text{ }^{ii} \quad (10)$$

As it can be observed, in this important scientific paper, there is no trace of the ability of this model to work in a negative interest rates environment, it is, however, from this fundament that my work wants to start.

2.2 A theory of the term structure of interest rates by John C. Cox, Jonathan E. Ingersoll Jr., and Stephen A. Ross

The research done by Cox, Ingersoll, and Ross "uses an intertemporal general equilibrium asset pricing model to study the term structure of interest rates".³

² Vasicek, O. A. "An equilibrium characterization of the term structure." *Journal of Financial Economics*.

³ Cox, J. C., Ingersoll, J. E. and Ross, S. A. "A theory of the term structure of interest rates." *Econometrica*, Volume 53-2.

Starting point: what is the term structure of interest rates? It measures the relationship among the yields on riskless securities that differ only in their term to maturity, explaining the market's anticipations of future events. It is necessary here to take into account three main hypothesis. However, before listing all of them, it is important to be focused on one crucial aspect of this model which is indeed linked to the proposed topic I decided to work on. As a matter of fact, the CIR model does not allow the use of negative interest rates in its application. This is why, nowadays, it is necessary to question these models and propose new ways of working.

The first assumption to be mentioned is the expectations one; which states that the bonds are priced in a way such that the implied forwards rates are equal to the expected spot rates. After only this introduction it is possible to derive two important postulates: first of all, the return provided by holding a long-term bond to maturity is equal to the expected return on repeated investment in a series of short-term bonds; then, the expected rate of return over the next holding period is equal for bonds of all maturities. Going ahead with the second hypothesis, it is important to talk of the liquidity preference one. According to it, forward rates are pushed to be always greater than expected spot rates by risk aversion, moreover, this difference between them represents the amount thanks to which investors are pushed to hold longer-term securities. Last but not least, the market segmentation hypothesis, according to which individuals are driven in their choices by strong maturity preferences, and there are different and separate markets in which it is possible to find bonds with different maturities. After the analysis of the model, it is possible to come up with two results. The equilibrium interest rate can be written as:

$$\begin{aligned}
 r(w, y, t) = \frac{\lambda^*}{WJ_w} &= a^{*'}\alpha + a^{*'}GG'a^*W\left(\frac{J_{ww}}{J_w}\right) + a^{*'}GS'\left(\frac{J_{wy}}{J_w}\right) \\
 &= a^{*'}\alpha - \left(\frac{-J_{ww}}{J_w}\right)\left(\frac{VarW}{W}\right) - \sum_{i=1}^k \left(\frac{-J_{wyi}}{J_w}\right)\left(\frac{CovWYi}{W}\right)
 \end{aligned} \tag{11}$$

The equilibrium value of any contingent claim, F , must satisfy the following differential equation:

$$\Phi_w F_w + \Phi_y F_y \quad (12)$$

This equation represents the risk premium for a security that is in equilibrium. To sum up, the bond prices depend only on one random variable, as in the Vasicek model, which serves as an instrumental variable for the underlying technological uncertainty.

$$P(r, t, T) = A(t, T)e^{-B(t, T)r} \quad (13)$$

This formula, which identifies the price of a bond, can be defined a decreasing convex function of the interest rate and maturity, and an increasing function of the time. Specifically, a decreasing convex function of the mean interest rate level θ , which is the long term rate at which r tends in the future, and of the speed of adjustment parameter κ if the interest rate is less than θ . Concluding, the dynamic of bond prices is given by:

$$\Delta P = r[1 - \lambda B(t, T)]Pdt - B(t, T)P\sigma\sqrt{r}dz_1 \quad (14)$$

which means that the returns on bonds are perfectly negatively correlated with changes in the interest rates.

2.3 Conclusion

To conclude this literature review, it is important to keep in mind and have a clear view on the main contributions and assumptions given by these studies. The most important thing to take into account is that all the financial models, including these two I have decided to study, are based on the main concept that in the reality it is not possible to have negative interest rates. This is not what has happened in the real world, since after the last economic crisis a lot of central banks have started to set interest rates with a negative sign in order to give the possibility to the economy of restarting and being re-boost. Given this framework, in which the state of the art is stopped in an environment different from the reality, the purpose of my study is to

analyze these models and figure out new developments in order to try to fit this literature to the new economic scenario, seeing if these models with the needed adjustments can work.

3. Implementation of the models

The purpose of this chapter is to show the implementation of the models reviewed in the previous paragraph. I decided to proceed with the study in three different and separate steps.

The first one, which is also the most important given the main goal of my research work, concerns the study of the Vasicek and CIR model to see if they can work in both economic scenarios of positive and negative interest rates. What it will come out from this analysis is that a shift in the conditions brings some needed and essential changes in the models, in order to be adapted to the new conditions of the market. Without generating these alterations, in fact, it is not possible, as I will show, to apply the CIR model, due to one of its fundamental assumption. After this first assessment and to confirm whether or not the two models provide the same results, it has been made a comparison between them, first calculating the bond prices with the estimated term structure and then making the same procedure but with the current-term structure, which is perfectly aligned with the market economy. In this part, the purpose has been to show the error produced by the model in the estimation of the term structure.

Finally, the last step of the development has been the comparison of the two models established in the first stage and the no-arbitrage one-factor model of Hull and White. The scope of this development has been to compute the trend of the error that comes out from the difference between the bond prices of Hull and White with both the two equilibrium models. In this case, the main objective of the work has been to present how much the two equilibrium models are far from being the better tool used by traders.

3.1 Analysis and construction of the first development

Besides every possible use and development of these models, the main purpose has been to question on the possibility for them to work in a reality that is by now so far from the one

studied in all the manuals. Each financial model is set on some important assumptions, which allows it to operate and provide the expected results. It is, therefore, impossible to disregard from them, and moreover it is essential go through them. In this case, while for Vasicek model the possibility of allowing negative interest rates is contemplated, even if never applied until now (since some years ago, negative interest rates never appeared in the economy); for CIR model there is the strict ban regarding this prospect; since, first of all, it is stated in one of the fundamental assumptions of the model, and proceeding, mathematically speaking, the formulas provided by the model are not able to make calculations when a negative sign appears. As a matter of fact, while for the former, the process of r is given (as it has already cited) by:

$$dr = a(b - r)dt + \sigma dz \quad (15)$$

for the latter, the process has a tiny difference, which though creates a relevant problem, i.e.:

$$dr = a(b - r)dt + \sigma\sqrt{r}dz \quad (16)$$

Having the \sqrt{r} in the last term of the equation, this does not allow the stochastic process for r to be calculated when the input of the model is a negative interest rate. In order to fix this problem, the idea has been to add a variable in the determination of r . Let's call this parameter α , and let's define r equals to the rate (given by x , which in the case should be the negative rate observed in the market, plus α).

$$r(t) = x(t) + \alpha \quad (17)$$

The parameter α needs to be adjusted every time, being equal at least to the maximum negative interest observed in the term structure and as it can be obviously understood greater than 0, in order to offset the negative sign of $x(t)$. Once this alteration has been made, the next step in the analysis has been the construction of the term structure for both the Vasicek and the CIR model. In both the cases, I used a Monte Carlo simulation to estimate the process for r , obtaining 500 dr trials. This procedure has been repeated for 10 years with a quarterly frequency.

The inputs given for this estimation have been $r(0m)$, the three months' rate observed at time 0; the parameters a (i.e. the speed of the rate adjustment), b (i.e. the mean-reverting level of the rate in the long-term), and σ (σ , i.e. the volatility linked to the term structure observed in each year); and the variable Δt (in this case chosen to be equal to 3 months). After completing the entire simulation, I took the average for each period and added this mean value of dr to the spot rate. Let's explain it e.g. for the rate I used as input at time 0 (i.e. $r(0m)$). Once I simulated dr for the first 3 months, and I took the average of this value, let's call it dr_1 I found $r(3m)$ as:

$$r(3m) = r(0m) + dr \quad (18)$$

An important part of the work, in this stage, has been the estimation of the parameters a , b , and σ used in the procedure previous explained. In order to perform this estimation, I used the time-series method; which is the procedure explained and tested by Hull in the last published edition of Options, futures, and other derivatives (the 10th edition), where it has been dedicated one paragraph precisely to the estimation of the parameter for the Vasicek model.

This procedure makes use of the historic time series of a zero rate and try to estimate the process pursued by r in the real world through a linear regression. The first thing has been to take the real world process for the short term rate on a daily basis and calculate the change as $(r_{t+1} - r_t)$. Subsequently, I have performed a regression of the change in the rates $(r_{t+1} - r_t)$ against the rates itself (r_t) , and a , b , and σ have been calculated from the regression results. Precisely, a has been computed as:

$$(-\text{coefficient of variable } X1 * 250) \quad (19)$$

b^* has been set equal to:

$$\left(-\frac{\text{coefficient of the intercept}}{\text{coefficient of the variable } X1}\right) \quad (20)$$

and σ as:

$$\text{standard error} * \sqrt{250} \quad (21)$$

Since it is possible to count about 250 observations per year, and Δt is $\frac{1}{250}$, the computations made above show how the data obtained from the regression need to be annualized. Until now, through this procedure, I have been able to calculate the parameters in the real world. Now, in order to transform them in risk neutral parameters, it is necessary to insert the market price of risk, named λ . Using a trial value of lambda, I proceed using the Vasicek model equations of $A(t, T)$, $B(t, T)$, (which can be found in the appendix ii), and the formula of the zero rate:

$$R(t, T) = -\frac{1}{T-t} \ln A(t, T) + \frac{1}{T-t} B(t, T) r(t) \quad (22)$$

In this way, I have obtained the zero-coupon rates as function of maturity and I had the possibility to compare it with the market zero-coupon rate. At this point, for the last stage, I used the solver to determine the value of λ that minimizes the sum of squared errors between the zero-coupon rates given by the model and those taken from the market. In this way, the real world parameters can be converted to risk-neutral parameters, they can be used to apply the formulas and to derive the bond prices. In order to confirm whether the parameters obtained are significant, it is possible to check the results using the maximum-likelihood method. As a matter of fact, this procedure consists in maximizing the likelihood function in relation to the values assumed by the parameters, which are the object of the estimation. In my case, all the estimations have been confirmed. The likelihood has been computed as function of different parameters:

$$Likelihood = \sum -\ln \left(\frac{\sigma^2}{250} \right) - \left(\Delta r - \left(\frac{a(b^* - r)}{250} \right) \right) \quad (23)$$

Switching now the focus on CIR model, I tried to apply the same procedure just mentioned above for Vasicek, but in this case the method does not work for the entire period taken into account since as I have already mentioned above, CIR model does not allow negative interest rates. Indeed, in the computation, when a negative interest rate appears, the result gives an error; therefore, the parameters can be computed only for a shorter period, in which rates are always

positive. However, in my specific and particular case, I decided to use the same parameters found for Vasicek also for CIR model, since this provides consistency when comparing the models between them. The only variation needed, from a structural point of view, has been the volatility, given that:

$$\sigma(vas) = \sigma(CIR) * \sqrt{r(0m)} \rightarrow \sigma(CIR) = \frac{\sigma(vas)}{\sqrt{r(0m)}} \quad (24)$$

Everything I have explained until this stage has been done to apply the two models and perform the first test I supposed to do, which consisted in demonstrating that a shift in the condition of the market brought some needed and essential changes in the models. Without generating these alterations, in fact, it should not be possible to apply the CIR model, due to one of its fundamental assumption. Proceeding with the stream of thought anticipated in the introduction of this chapter, now I will introduce the second part of the analysis, which concerns the comparison between the two equilibrium models applied first with the estimated term structure and then with the current-term structure. In this way taking all the other parameters equal to the previous step (i.e. *ceteris paribus*), the purpose has been to analyze the error provided by the models. Particularly, how much these equilibrium models stand apart from the real world. Indeed, using the same models and changing only the input (i.e. the different term structure), the result produced has been the error given by the estimation of the term structure, in other words, how far it stands from the current one. The difference with respect to the first part of the research has been the absence of the computation of the process for r . Especially, I decided to take one of the main assumption of the no-arbitrage model, which is having the entire current-term structure as an input of the model and put it in both the equilibrium models, to give an outlook of the error committed when pricing bonds with these one factor equilibrium models. Finally, in the third stage of my analysis has been introduced another model, the no-arbitrage one factor model of Hull and White. The purpose has been to make a second comparison also this time with the two models studied above, which are the main topic of my research. Here,

the work has been constructed confronting the Vasicek and CIR model, applied as they have been developed (i.e. with the estimated term structure) with the Hull and White model, which expects the computation of bond prices through the current-term structure. As a matter of fact, since this is considered the input of the model, there is the perfect fit with the real world data. Moreover, in this model it is necessary to introduce also the parameter λ , which is needed to determine the parameters $\theta(t)$ and $\phi(t)$, which then are linked to the computation of $A(t, T)$. In order to compute this parameter, I used the VBA tool on excel. Since λ is a constant that should be fixed by the user, I decided to use the goal seek command planned with a macro, with the aim to find the value of $\lambda(t)$ that makes $A(t, T)$ equals to the ones found applying Vasicek and CIR with the real term structure (this has been done for simplicity and consistency reasons). Obviously also in this case, the other parameters (which are a , b and σ) have been taken equal to the ones I have calculated in the first part, with the time-series procedure explained and used by Hull, for consistency purpose. This time, the main result I expected from the analysis was not as before a proper study of the error committed by the two equilibrium models in estimating the term structure, but more accurately a real investigation on the operation of these models in pricing bonds against the no-arbitrage model.

Concluding, I would like to highlight that once the entire analysis has been structured I proceeded testing the procedure for some different years, first for that years, going back in the past, where interest rates were positive and right away for the more recent years in which negative interest rates have occurred. I will show all these results in the third and last paragraph of this chapter, only after having explained which data I have used to perform all the model just proposed.

3.2 Data

The data used for the computation in the different steps of the analysis have been taken from different sources and comprehend different indexes of the European market.

First of all, in order to proceed with the application of the time-series method, I used the OIS zero rates at 1 month, which are market rates and not theoretical ones. The choice has been driven by the features of this rate. OIS, indeed, are popular among financial institutions because they are considered a good indicator of the interbank credit market and less risky than other well-known interest rates. Moreover, they are generally short-term rates, and this is useful to see the evolution of the yield curve during the years. The time frame taken has been from September 2010 until February 2017, with 1631 observations used for performing the linear regression of the change in rates against the rates itself. Looking at the data I kept, it is possible to see how the rates have been lowered increasingly between 2013 and 2014, up to becoming negative precisely on 1st September 2014 ($r = -0.01\%$). Once the regression has been performed, as I have already explained in the first paragraph, the output generated consisted in the real world parameters a , b^* , and σ .

$$a = 0.1772; b^* = -0.00575; \text{ and } \sigma = 0.279\%$$

The following step has been the conversion of the real world parameters into risk neutral ones. The method used has been the one explained before, which consists in the use of the solver in order to minimize the sum of squared differences between the rates observed in the market and the one estimated by the model. For what concerns the market rates, I took the EONIA⁴, which is the daily interbank interest rates. I decided to use this rate since it is the one I took also as input for the estimation of the term structure in the subsequent steps. Then, I computed the sum of squared differences between these rates and the one provided by the Vasicek Model for a trial lambda. Finally, from the minimization of this sum I found the optimal value of λ to transform the real world parameter in the risk neutral one. Precisely, a and σ have kept their initial value, while b^* has been changed in b .

⁴ Bloomberg

$$b = \frac{b^* - \lambda * \sigma}{a} \quad (25)$$

In the table reported below, there are all the parameters found for each year of the study. The method has been always the same for all the years taken into consideration, what has changed every time has been the input $r(0m)$, while the regression has been based always on the same set of data mentioned above (which therefore includes all the years studied).

		Risk-neutral					
	Real-world	2010	2011	2012	2015	2016	2017
a	0.1772	0.1772	0.1772	0.1772	0.1772	0.1772	0.1772
b*	-0.575%	6.245%	6.204%	6.979%	1.810%	2.587%	2.147%
σ	0.279%	0.279%	0.279%	0.279%	0.279%	0.279%	0.279%
Trial λ		-4.3272	-4.3010	-4.7929	-1.5130	-2.0065	-1.7269

Here, one thing I noticed has been that the trials λ are smaller than the one estimated by Ahmad and Wilmott for the long term period and for the American market (i.e. $\lambda = -1.2$)⁵. This demonstrates significance and consistency, first because of the different market, the European one, which is riskier; and also due to the fear factor that has appeared when the conditions of the market became critical (as it has been during these years taken for the analysis). The same data used for the linear regression (OIS rates) have been also used for the check made with the maximum-likelihood method, which has provided the same results just showed above. Proceeding with the next stage of the analysis, now it is the time to speak about the proper application of the two equilibrium models. In this part, I have inserted in the models the parameters found above, using the same constants of Vasicek also in CIR. What I have added, this time, has been the instantaneous rate $r(0m)$. I used the EONIA rates related to each year, and taken from Bloomberg.

2010	2011	2012	2013	2014	2015	2016	2017
0.34%	0.42%	0.39%	0.06%	0.15%	-0.08%	-0.24%	-0.36%

⁵ Hull, John. 2014. *Options, futures, and other derivatives*. Boston: Prentice Hall.

From these data, I started the computation of the process for r , following the process explained in the paragraph one through the use of Monte Carlo simulation. Let's take for example the EONIA rate for January 2010, i.e. 0.34%. For that year, I simulated the process of dr using the formula $dr = a(b - r)dt + \sigma dz$, where dz is the stochastic term, with $a=0.177$, $b=6.34\%$, $\sigma_{\text{vas}}=0.28\%$, and $\Delta t=3 * \frac{30}{360}$. Once, I repeated this computation for 500 trials, I took the average and I added it to the rate of the EONIA 0-month rate.

$$r(3m) = r(0m) + \text{average}(dr) \rightarrow 0.60\% = 0.34\% + 0.26\%$$

The same example can be given for one year with a negative input, let's take year 2017 with a 0-month EONIA of -0.36%, $a=0.177$, $b=2.15\%$, $\sigma_{\text{vas}}=0.279\%$, and $\Delta t=3 * \frac{30}{360}$. This time, I had:

$$r(3m) = r(0m) + \text{average}(dr) \rightarrow -0.24\% = -0.36\% + 0.12\%$$

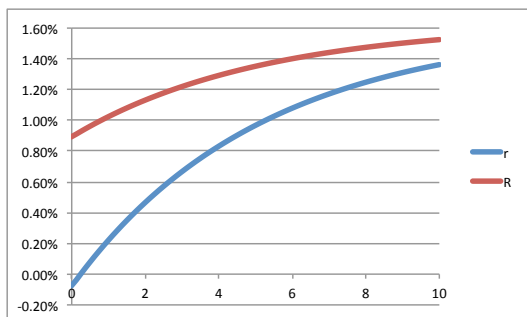
At this point, when willing to estimate the error produced by the two models in the estimation of the term structure and for the comparison between them and a no-arbitrage model, I needed another term structure observable in the market. Since, the OIS and EONIA are only instantaneous and short term rate, I decided to opt for the ECB yield curve, which is a theoretical term structure estimated with some financial models, but in my opinion the closest to the current term structure, I would call it the best available alternative observable in the market. The data to which I am referring have been taken from ECB website⁶. There is, in fact, a section called "Statistics", in which it is possible to find all data related to Euro area yield curves under the tag "Financial markets and interest rates". The yield curves selected are the ones including all euro bonds and not only the AAA rated bonds. For what concerns the time frame, I kept 3 years with positive interest rates (precisely January 2010, January 2011, and January 2012) and three years with negative sign rates (March 2015, January 2016, and January 2017).

⁶https://www.ecb.europa.eu/stats/financial_markets_and_interest_rates/euro_area_yield_curves/html/index.en.html

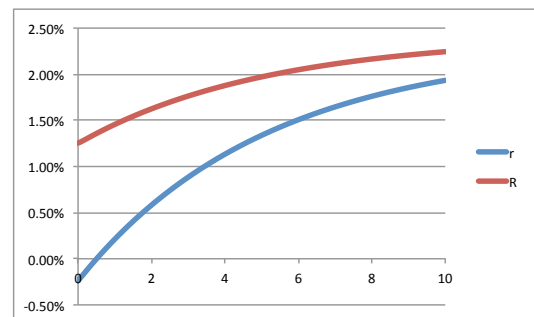
At first I have used this curve to examine how much it deviates from the one I have forecasted, and hereafter I used it in the no arbitrage model of Hull and White.

3.3 Results

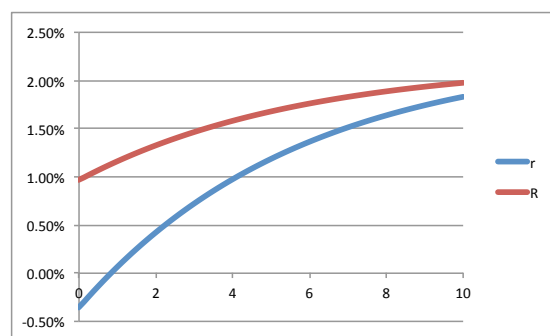
In this last paragraph related to the explanation of the model, I would like to provide some insights into the results I have obtained. Following the same structure of the passage 3.1, I would like to start from the ground, talking about the application of the models in presence of negative interest rates. As I explained above, it has been possible thanks to some adjustments and here I would like to display, to prove their application, the term structures obtained for the three negative interest rates years (respectively 2015, 2016, and 2017). As I have already explained, since for Vasicek model there were no problems (in fact it contemplates the possibility to have negative interest rates), I have worked a little bit on the adjustments needed for CIR, and for this reason I would like to show results related to it.



7



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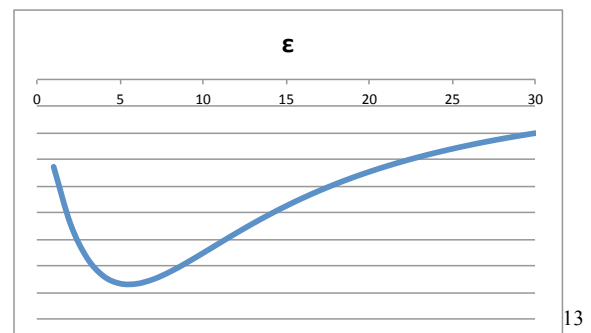
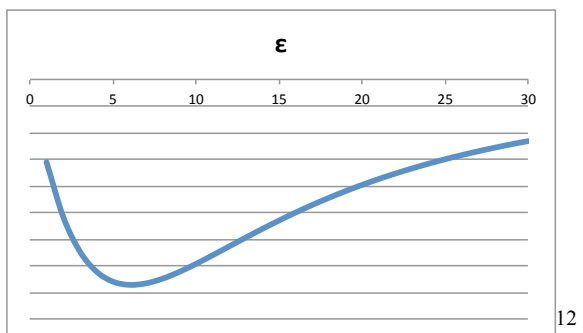
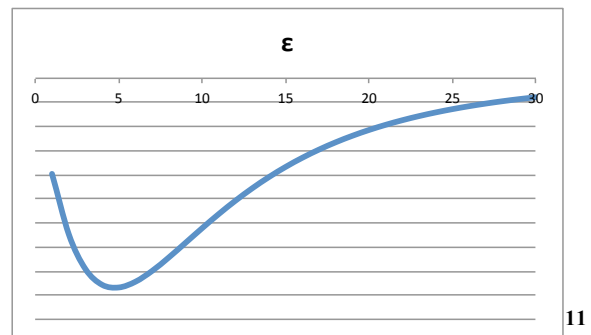
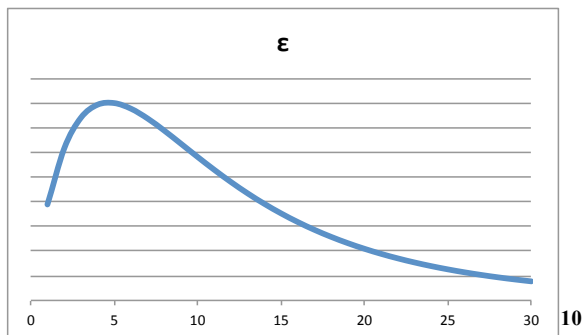
9

⁷ Graph 1: CIR process for r and R in 2015

⁸ Graph 2: CIR process for r and R in 2016

⁹ Graph 3: CIR process for r and R in 2017

As it is shown in the graphs and as I have explained in the first part of this chapter the models can be applied even when the $r(0m)$, input of the model, has a negative sign. As it is highlighted looking at the EONIA rates, we have in 2015, 2016, and 2017 respectively -0.08% , -0.24% , and -0.36% . Another thing I would like to bear in mind is that the term structure of CIR tends to lie always under the yield curve provided by the Vasicek model, in both the scenario of positive and negative interest rates, when these two are put in comparison. Going ahead with the second part of the study I would like now to show the trend of the error estimated between the price of the bonds calculated through the two equilibrium models, first, with the term structure originated by the Monte Carlo Simulation, and then, in a second moment, with the current-term structure, perfectly fitted with the reality. In this way, as it has been explained in the previous paragraph, it is possible to measure the error done by the model in the estimation of the term structure (i.e. $R_{vas} - R_{MKT}$).



¹⁰ Graph 4: ϵ for Vasicek model in 2010

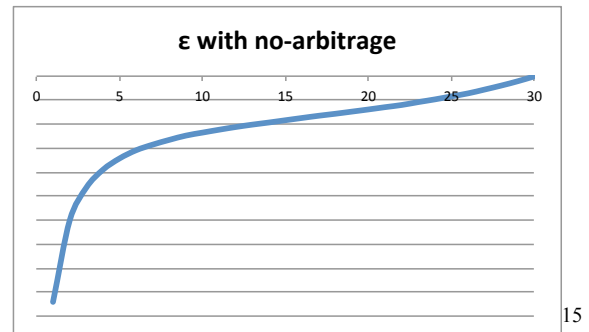
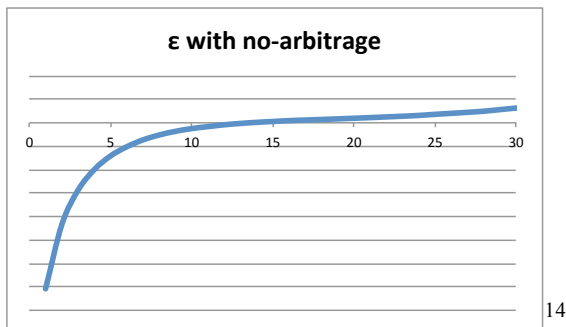
¹¹ Graph 5: ϵ for CIR model in 2010

¹² Graph 6: ϵ for Vasicek model in 2015

¹³ Graph 7: ϵ for CIR model in 2015

The four error trends proposed above are only some examples I have decided to put here (since for space problem it is not possible to display all the graphs obtained in the study). Precisely, these are referred to year 2010, and 2015, to provide a view on both scenario of positive and negative interest rates, and for both the models of Vasicek and CIR. As it can be observed from the graphs, the trend found has been always the same, with a huge perception of the error always between 0 and 20 years (with a peak circa at 7 years). What is also important to mention, in my opinion, is that the two equilibrium models, which were already not reliable with positive interest rates, seem to be even less reliable when negative interest rates appear, giving a higher level of the estimation of the error. For this reason, it is well-known among traders that these approaches are not satisfactory; as a matter of fact, they believe it is not possible to trust in the bond price when it is computed with one of these models. Finally, for the last part of the study I would like to show the error estimation given by the difference between the equilibrium model with the “fictitious” term structure and the no-arbitrage model of Hull & White, where, as it has been already said, there is perfect fit with the real world term structure. From the previous analysis, it has seemed that in the long term the process followed by r estimated with a Monte Carlo simulation can be considered a quite good estimation of the real term structure, while in the short term the error is quite big. The difference from the previous analysis is that in the first case the error is in absolute value, meaning that it is given by the application of the same models with two different term structures (precisely, one estimated and one that is the current-term structure); instead in this case, the error is more related to the outcome of the two different models, i.e. the prices of the bond (i.e. while in the first scenario I have applied the same model, in the latter one I have compared two different models based on different assumptions).

Also for this last part of the analysis, I have displayed only two years for space purpose, but overall the trend studied has been pretty equal and homogeneous for all the years taken into consideration.



As a matter of fact, as it has been found in the previous results obtained in the second stage of the research, the two equilibrium models seem to generate a huge problem in the first years, while going ahead in the time, they converge towards the same results obtained applying the Hull and White no arbitrage model.

3. Conclusions

It is clear, after all this analysis, that reality has changed and financial models need to be adapted in order to fit this new scenario and work in all possible directions. This is an important and crucial objective for economists, since this trend in interest rates affects monetary policies and since this choice of central banks has been made to give renaissance to the economy, it influences also citizens making impossible savings, but boosting on borrowings. What will be the effect on private savings and why does a bank have to pay the central bank to keep its deposit, losing earnings? Which are the positive sides in this emblematic decision? Mario Draghi, the Italian governor, together with the ECB board, has tried to transmit trust and calmness, explaining that there will be no direct impact on citizens' savings, even if the effect could be indirect. The most hit agents are commercial banks, which have to pay ECB to keep their money stalled instead of using it to lend money and give breath to the market. This has been, really, the expectation on which this decision has been based. As a matter of fact, ECB had a positive hope that the banks stopped accumulating money and started lending more to

¹⁴ Graph 8: ϵ between Vasicek model and Hull & White in 2010

¹⁵ Graph 9: ϵ between Vasicek model and Hull & White in 2015

consumers, businesses, or among banks, boosting the economy.¹⁶ However, as each medal that is respected, there is the downside, which in this case is identified in the willingness of banks to transfer this major cost on customers, bringing again the economy at a stagnant point. Apart from this negative side, ECB's purpose was conceived in order to create an environment in which savers should be seen as the "hero" agents; as a matter of fact, once the monetary accommodation will be reached, they will be the ones identified as supporters of growth and as a foundation for the increase in rates. Given this concept, here, it is spontaneous questioning the reason behind the "punishment" of savers, given their important role, and instead the reward for borrowers. Fortunately, also this time, ECB has immediately clarified its position. As a matter of fact, its core business is making more or less attractive for households and businesses to save or borrow money, but this is not done in the spirit of punishment or reward.¹⁷

Monetary policy makers often think in terms of a concept known as the real equilibrium rate or the "natural" rate of interest, the rate which is consistent and coherent with the level of the inflation in the economy. Setting short-term interest rates below this rate has the effect of pushing upward economy and inflation, with the hope of reaching the target imposed to ECB by the Maastricht agreement. This is the reason why the bigger central banks in the world, among which ECB, have started handling accommodative monetary policies.

From a different point of view, there are, as usual, downsides concerning this new macroeconomic scenario. First of all, private citizens and businesses, in the long term, will prefer to retire their deposits, preferring to hold cash on which they have not to pay any interest (of course with other risks annexed, such as the risk of being stolen). Then, another issue regards the way people used to value things, i.e. in nominal terms rather than in real ones. The belief that bigger is better takes shape also and especially in this case. As a matter of fact, people used

¹⁶ Debanjan, D. "Negative Interest Rate Policy by ECB: A Case Study." *Skyline Business Journal*, Volume X.

¹⁷ European Central Bank. "The ECB's negative interest rate." <https://www.ecb.europa.eu/explainers/tell-me-more/html/why-negative-interest-rate.en.html>

to look at negative interest rates as an unnatural event. Last issue, negative rates can be linked to something irrational because of institutional problems and the lack of knowledge on tax and legal discipline on this subject. Arrived at this point, after the initial promise of only few months of negative signs, but given the extended period, it is necessary and essential ask some important questions on this situation. First, how much lower can we go? And, do the persistence of low and/or negative interest rates pose particular challenges to the stability of the financial system?¹⁸ For the first question, it has been explained that the lower bound, named reversal rate, is charged taking into account the opportunity cost of holding cash and trying to balance bank's benefits and costs .

Everyone is asking: will these policies push the economy, re-giving life to Europe and generally to developed countries, or will it worsen the already dangerous reality?

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¹⁸ European Central Bank. "Assessing the implications of negative interest rates." <https://www.ecb.europa.eu/press/key/date/2016/html/sp160728.en.html>

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European Central Bank. "Assessing the implications of negative interest rates." <https://www.ecb.europa.eu/press/key/date/2016/html/sp160728.en.html>

Appendix

$$^i \mu(t, s, r) = \frac{1}{P(t, s, r)} \left[\frac{\partial}{\partial t} + f \frac{\partial}{\partial r} + \frac{1}{2} \rho^2 \frac{\partial^2}{\partial r^2} \right] P(t, s, r), \text{ and } \sigma(t, s, r) = -\frac{1}{P(t, s, r)} \rho \frac{\partial}{\partial r} P(t, s, r)$$

$$^{ii} B(t, T) = \frac{1 - e^{-a(T-t)}}{a} \text{ and } A(t, T) = e^{\frac{[B(t;T)-T+t](a^2 b - \frac{\sigma^2}{2})}{a^2} - \frac{\sigma^2 B(t;T)^2}{4a}}$$

$$^{iii} B(t, T) = \frac{2(e^{\gamma(T-t)} - 1)}{(\gamma + \kappa + \lambda)(e^{\gamma(T-t)} - 1) + 2\gamma}, \text{ where } \gamma = [(\kappa + \lambda)^2 + 2\sigma^2]^{\frac{t}{2}} \text{ and}$$

$$A(t, T) = \left[\frac{2\gamma e^{\left[\frac{(\kappa + \lambda + \gamma)(T-t)}{2}\right]}}{(\gamma + \kappa + \lambda)(e^{\gamma(T-t)} - 1) + 2\gamma} \right]^{\frac{2\kappa\theta}{\sigma^2}}$$